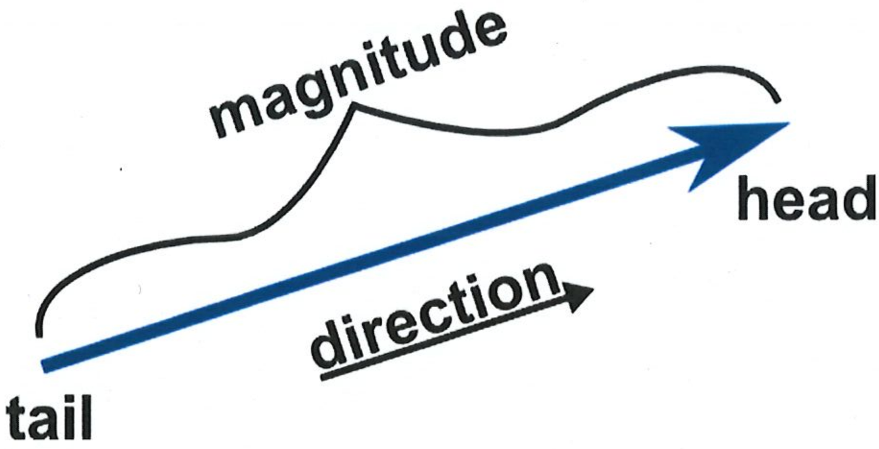
Chapter 0: Introduction of Vector

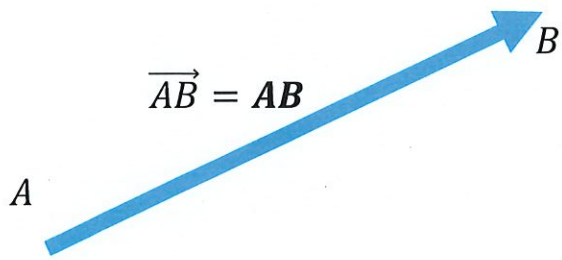
Vector

A vector is a quantity that possesses a direction and a magnitude in plane (in 2 dimension) / space (in 3 dimension). Sometimes, we may use a (directed) line segment to represent the vector



The arrow on the lines indicates the intended direction of the vector and the length represents the magnitude of the vector. The magnitude is also named the **modulus** or the **length** of the vector.

Consider an example that the directed line segment represents the vector $\overrightarrow{AB}$ where the arrow on the figure represents the direction from A to B and the length of the line AB represent the magnitude of the vector. Note that vector sometimes is presented in boldface, e.g. $\overrightarrow{AB} = \boldsymbol{AB}$. In this course, we will use both presentations interchangeably.



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1. Magnitude (Modulus)

The magnitude of a vector $\overrightarrow{AB}$, written as $|\overrightarrow{AB}|$, is equal to its length.

In particular, when a vector has zero magnitude, it is called zero vector, $\vec{0}$.

For example,

If $\overrightarrow{AB}$ is a displacement (vector), then $|\overrightarrow{AB}|$ is the distance.

If $\overrightarrow{AB}$ is a velocity (vector), then $|\overrightarrow{AB}|$ is the speed.

2. Unit vector

A unit vector is a vector with magnitude equal to 1. In order to obtain a unit vector in the direction of a vector, say $\vec{a}$, we divide by its magnitude.

To indicate a vector is a unit vector, we give a 'hat' to the vector, denoted as

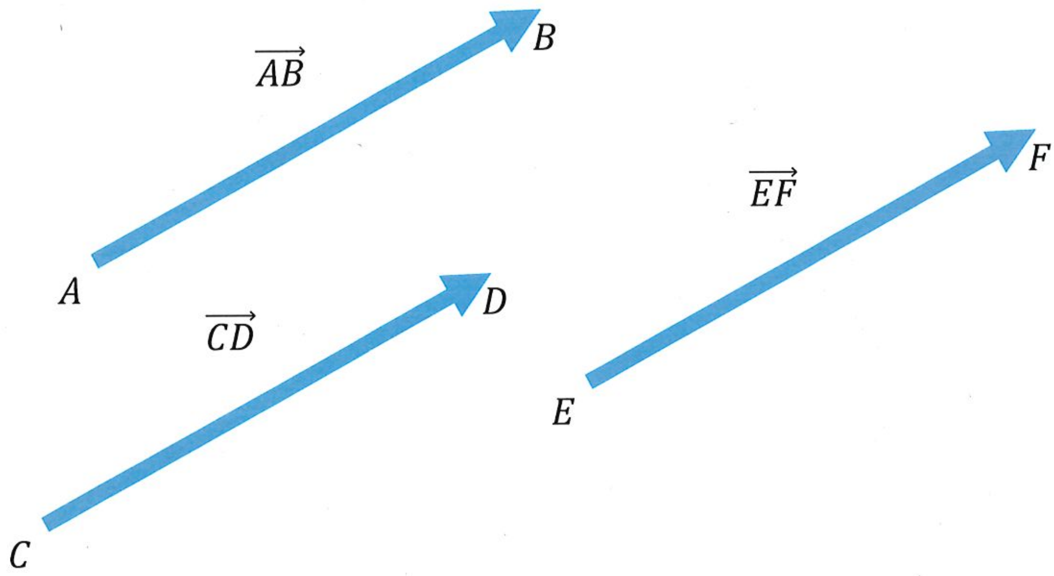
$$\hat{\vec{a}}$$

and it can be expressed as

3. Equal vector

Two vectors are said to be equal if they have the **same length and direction**.

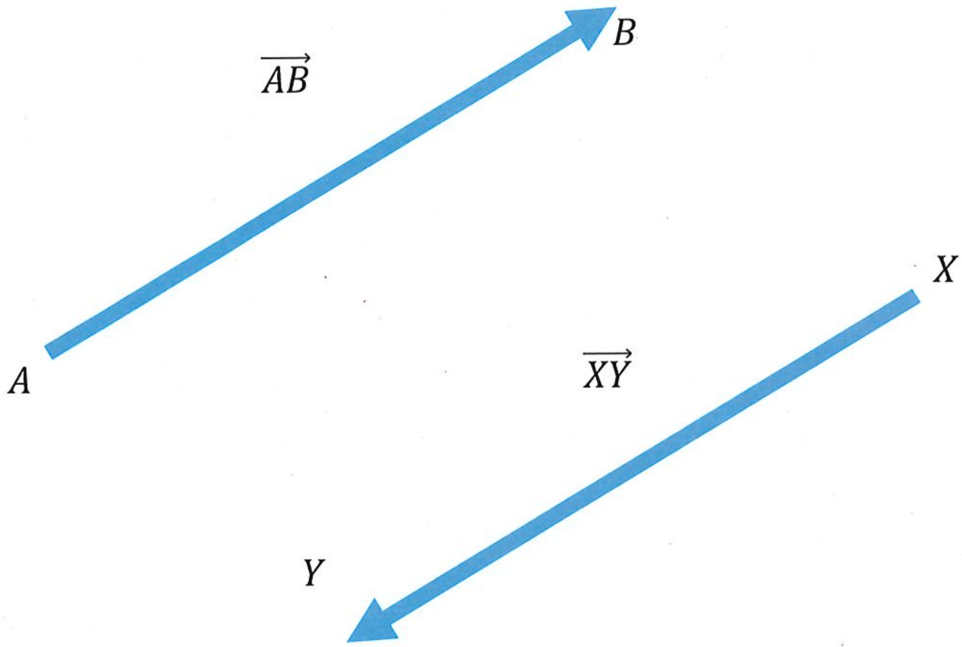
Consider three vectors $\overrightarrow{AB}, \overrightarrow{CD}$ and $\overrightarrow{BF}$ below. Each of them is parallel to each other and they possess the same length, i.e. $\overrightarrow{AB} = \overrightarrow{CD} = \overrightarrow{EF}$.



It implies that vectors can be **moved**!

4. Negative Vector

$\overrightarrow{XY}$ is a vector having the same magnitude as the vector $\overrightarrow{AB}$ but it is pointing to opposite direction of $\overrightarrow{AB}$. We call $\overrightarrow{XY}$ to be the negative $\overrightarrow{AB}$, i.e. $\overrightarrow{XY} = -\overrightarrow{AB}$ or $\overrightarrow{AB} = -\overrightarrow{XY}$.



5. Scalar Multiplication

Multiplying a vector by a non-zero number (scalar) λ results in another vector.

If $\vec{a}, \vec{b}$ are two vectors such that $\vec{b} = \lambda\vec{a}$ where λ is a non-zero number, then $\vec{a}$ and $\vec{b}$ are **parallel** vectors.

(i) If $\lambda > 0$, then $\vec{a}, \vec{b}$ have the <u>same</u> direction.

(ii) If $\lambda < 0$, then $\vec{a}, \vec{b}$ are in <u>opposite</u> direction.

6. Vector Addition and Subtraction

General Vector Operations

i. $\boldsymbol{a} + \boldsymbol{b} = \boldsymbol{b} + \boldsymbol{a}$

ii. $(\boldsymbol{a} + \boldsymbol{b}) + \boldsymbol{c} = \boldsymbol{a} + (\boldsymbol{b} + \boldsymbol{c})$

iii. $(m + n)\boldsymbol{a} = m\boldsymbol{a} + n\boldsymbol{a}$

iv. $m(\boldsymbol{a} + \boldsymbol{b}) = m\boldsymbol{a} + m\boldsymbol{b}$

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Example 1

A student drives 6 km due North before taking a right turn and then driving 6 km to the East. Lastly, the student makes a left turn and travels further 2 km to the North. What is the magnitude of the overall displacement of the student?

7. Position vector

We can use a **position vector** to tell the position (location) of one object (e.g. point) relative to another reference point. Specifically, a position vector is:

A vector which indicates the location or position of a given point with respect to an arbitrary reference point, the origin.

Often, vectors that start at the origin and terminate at any arbitrary point are called position vectors. These are used to determine the position of a point with respect to the origin. The direction of the position vector points from the origin towards the given point.

Given a Cartesian coordinate system in which we have a fixed origin O and a point $Q(x, y)$, the vector $\overrightarrow{OQ}$ is said to be the position vector of the point Q (with respect to the origin).

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Vectors $\vec{\imath}$ and $\vec{\jmath}$ are the unit vectors in the direction of the positive x and y axes in the 2 dimensional plane respectively. Clearly, they are perpendicular to each other.

From the figure, the vector $\overrightarrow{OR}$ is $x\vec{\imath}$ and the vector of $\overrightarrow{OS}$ is $y\vec{\jmath}$. The vector $\overrightarrow{OQ}$ is the addition of the vectors $\overrightarrow{OR}$ and $\overrightarrow{OS}$,

$$\overrightarrow{OQ} = \overrightarrow{OR} + \overrightarrow{OS} = x\vec{\imath} + y\vec{\jmath}$$

When we know the coordinates of a point, we can also use angular bracket to represent a vector, for example

$$\overrightarrow{OQ} = \langle x, y \rangle$$

When it comes to vector(s) with respect to $\vec{\imath}$ and $\vec{\jmath}$, if $\vec{q} = \langle x, y \rangle$, it represents the position vector of $Q(x, y)$, it has some properties of the unit vectors $\vec{\imath}$ and $\vec{\jmath}$:

i. $|\vec{q}| = OQ = \sqrt{x^2 + y^2}$

ii. Unit vector of $\vec{q}$

$$\hat{\vec{q}} = \langle \frac{x}{\sqrt{x^2 + y^2}}, \frac{y}{\sqrt{x^2 + y^2}} \rangle$$

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Suppose there are two vectors

$$\vec{u} = u_1\,\vec{i} + u_2\,\vec{j} = \langle u_1, u_2 \rangle \text{ and } \vec{v} = v_1\,\vec{i} + v_2\,\vec{j} = \langle v_1, v_2 \rangle,$$

the vector operations (addition, subtraction and scalar multiplication) can be taken place

i. Addition and subtraction

$$\vec{u} \pm \vec{v} = (\,u_1 + v_1)\,\vec{i} \pm (u_2 + v_2)\vec{j} = \langle u_1 \pm v_1, u_2 \pm v_2 \rangle$$

ii. Scalar multiplication

$$m\vec{u} = \langle mu_1, mu_2 \rangle$$

Example 2

Suppose that a plane is steered northeast, at an air speed of $400 \mathrm{mph}$, while the wind is blowing westward at $100 \mathrm{mph}$. To compute the ground speed of the plane, which accounts for both the steering and the wind, we need to add the velocity vectors corresponding to the plane and the wind. This yields a resultant velocity vector, whose magnitude is the ground speed.

To perform the addition of the velocity vectors, we need their components. For the plane's vector, the magnitude is 400 , and its direction is identified by an angle of $\pi/4$ radians with the positive x-axis, corresponding to northeast. It follows that the components of this vector are

$$\boldsymbol{v} = 400\left\langle \cos\frac{\pi}{4}, \sin\frac{\pi}{4}\right\rangle = 400\left\langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right\rangle = \langle 200\sqrt{2}, 200\sqrt{2}\rangle$$

Similarly, the wind's velocity vector, which makes an angle of π radians with the positive x-axis, corresponding to west, is

$$\boldsymbol{w} = 100\langle \cos\,\pi, \sin\,\pi\rangle = 100\langle -1,0\rangle = \langle -100,0\rangle$$

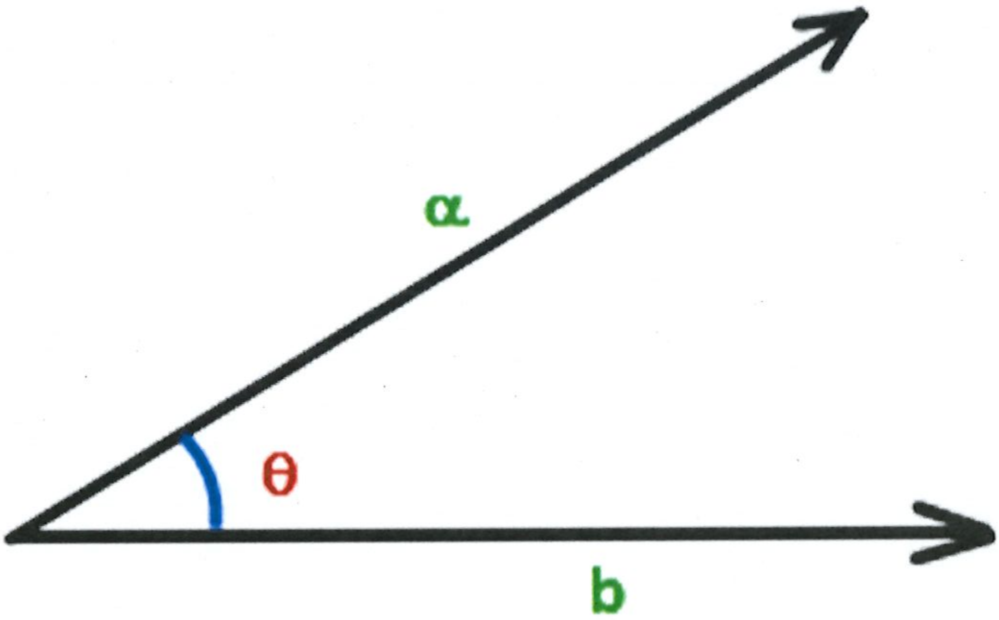
We now perform the addition to obtain the plane's true course (the true course or track of the plane is the direction of the sum of velocity vectors of the plane and of the wind) :

$$\boldsymbol{c} = \boldsymbol{v} + \boldsymbol{w}$$

To obtain the ground speed (ground speed is the length or magnitude of this sum of velocity vectors), we compute its magnitude, and obtain $|\boldsymbol{c}|$

8. Dot (Scalar) Product

Given two vectors $\vec{a} = \boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j} = \langle a_1, a_2 \rangle$ and $\vec{b} = \boldsymbol{b} = \langle b_1, b_2 \rangle = b_1\boldsymbol{i} + b_2\boldsymbol{j}$ in $\mathbb{R}^2$, let θ (where $0° \le \theta \le 180°$) be the angle between these two vectors.



Source: https://www.aplustopper.com/dot-product/

The dot product can be obtained by the following ways:

$$\boldsymbol{a} \cdot \boldsymbol{b} = |\vec{a}||\vec{b}| \cos\theta = a_1 b_1 + a_2 b_2$$

Remark:

i. Dot product will return a scalar (number).

ii. $\vec{\imath} \cdot \vec{\imath} = \vec{\jmath} \cdot \vec{\jmath} = 1$

iii. We say two vectors $\boldsymbol{a}$ & $\boldsymbol{b}$ are perpendicular (or orthogonal) if $\boldsymbol{a} \cdot \boldsymbol{b} = 0$.

iv. $\boldsymbol{a} \cdot \boldsymbol{a} = |\boldsymbol{a}|^2$ = square of its magnitude.

v. $\boldsymbol{a} \cdot \boldsymbol{b} = (a_1\vec{\imath} + a_2\vec{\jmath}) \cdot (b_1\vec{\imath} + b_2\vec{\jmath})$

vi. $\boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{b} \cdot \boldsymbol{a}$ (Commutative)

vii. $\boldsymbol{a} \cdot (\boldsymbol{b} + \boldsymbol{c}) = \boldsymbol{a} \cdot \boldsymbol{b} + \boldsymbol{a} \cdot \boldsymbol{c}$ (Distributive)

viii. $(k\boldsymbol{a}) \cdot \boldsymbol{b} = k(\boldsymbol{a} \cdot \boldsymbol{b}) = \boldsymbol{a} \cdot (k\boldsymbol{b})$ for any scalar k.

Example 3

The dot product of $\boldsymbol{u} = \langle 1,1 \rangle$ and $\boldsymbol{v} = \langle 2,-1 \rangle$ is $\boldsymbol{u} \cdot \boldsymbol{v} = 1 \cdot 2 + 1 \cdot -1 = 1$.

To find the angle θ between $\boldsymbol{u}$ & $\boldsymbol{v}$, we compute

$$\cos\theta = \frac{\boldsymbol{u} \cdot \boldsymbol{v}}{|\boldsymbol{u}||\boldsymbol{v}|} = \frac{1}{\sqrt{2}\sqrt{5}} = \frac{1}{\sqrt{10}}$$

which yields $\theta \approx 71.6$ degrees.

Example 4

Let $\boldsymbol{u}$ and $\boldsymbol{v}$ be vectors such that $|\boldsymbol{u}| = 3, |\boldsymbol{v}| = 4$, and the angle between them is $\pi/3$ radians, or 60 degrees. Then

$$\boldsymbol{u} \cdot \boldsymbol{v} = 3(4)\cos\frac{\pi}{3} = 12(\frac{1}{2}) = 6$$

Example 5

Let $\boldsymbol{u} = \langle\alpha, \beta\rangle$ be any nonzero vector. Then a vector that has the same length as $\boldsymbol{u}$, and is orthogonal to $\boldsymbol{u}$ is $\boldsymbol{v} = \langle\beta, -\alpha\rangle$. To verify this, we compute

$$\boldsymbol{u} \cdot \boldsymbol{v}$$

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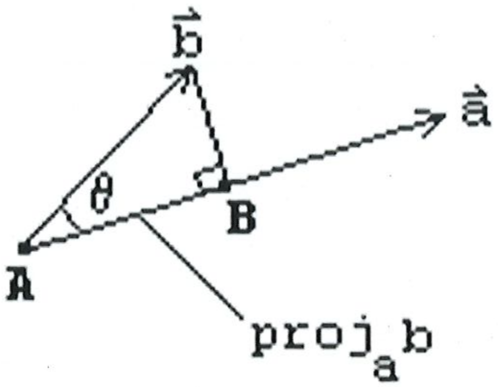
One important use of dot products is vector projection. The scalar projection of $\boldsymbol{b}$ onto $\boldsymbol{a}$ is the *length* of the segment AB shown in the figure below. The vector projection of $\boldsymbol{b}$ onto $\boldsymbol{a}$ is the *vector* with this length that begins at A points in the same direction (or opposite direction if the scalar projection is negative) as $\boldsymbol{a}$.



Source:http://sites.science.oregonstate.edu/math/home/programs/undergrad/CalculusQuestStudyGuides/vcalc/dotprod/dotprod.html

Thus, mathematically, the scalar projection of $\boldsymbol{b}$ onto $\boldsymbol{a}$ is $||\boldsymbol{b}|\cos\theta\,|$

(where θ is the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$) and can be written as

$$|comp_{\boldsymbol{a}}\boldsymbol{b}| = ||\boldsymbol{b}|\cos\theta\,| = |\frac{\boldsymbol{a}\cdot\boldsymbol{b}}{|\boldsymbol{a}|}|$$

Note that $\cos\theta < 0$ if $90° < \theta < 180°$.

This quantity is regarded as the component of $\boldsymbol{b}$ in the direction of $\boldsymbol{a}$ (hence the notation "$comp_{\boldsymbol{a}}\boldsymbol{b}$").

And, the vector projection is the unit vector $\frac{\boldsymbol{a}}{|\boldsymbol{a}|}$ multiplied by the scalar projection of $\boldsymbol{b}$ onto $\boldsymbol{a}$:

$$proj_{\boldsymbol{a}}\boldsymbol{b} = |\frac{\boldsymbol{a}\cdot\boldsymbol{b}}{|\boldsymbol{a}|}|\frac{\boldsymbol{a}}{|\boldsymbol{a}|} = |\frac{\boldsymbol{a}\cdot\boldsymbol{b}}{|\boldsymbol{a}|^2}|\boldsymbol{a}$$

Note that the scalar projection of $\boldsymbol{b}$ onto $\boldsymbol{a}$ is the magnitude of the vector projection of $\boldsymbol{b}$ onto $\boldsymbol{a}$, i.e.

$$comp_{\boldsymbol{a}}\boldsymbol{b} = |proj_{\boldsymbol{a}}\boldsymbol{b}|$$

Example 6

We can show that $\boldsymbol{b} - proj_{\boldsymbol{a}}\boldsymbol{b}$ is orthogonal to the vector $\boldsymbol{a}$.

Consider

$$\boldsymbol{a}\cdot(\boldsymbol{b} - proj_{\boldsymbol{a}}\boldsymbol{b})$$

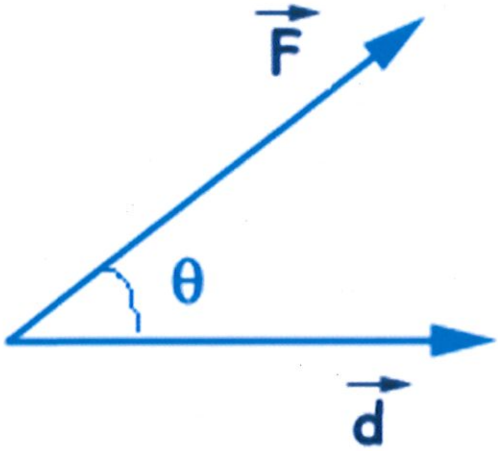
$$= \boldsymbol{a}\cdot\boldsymbol{b} - \boldsymbol{a}\cdot proj_{\boldsymbol{a}}\boldsymbol{b}$$

$$= \boldsymbol{a}\cdot\boldsymbol{b} - \boldsymbol{a}\cdot\left|\frac{\boldsymbol{a}\cdot\boldsymbol{b}}{|\boldsymbol{a}|^2}\right|\boldsymbol{a}$$

$$= 0$$

Example 7

Suppose that a box is pulled 10 m along the ground by a constant force of 50 N that is applied at an angle of $30°$ above the horizontal. Then, the horizontal force can be obtained by computing scalar projection of the force vector $\boldsymbol{F} = 50\langle\cos 30°, \sin 30°\rangle$ onto the displacement vector $\boldsymbol{d} = 10\langle 1,0\rangle$:



$$comp_d\boldsymbol{F} = \frac{50\left\langle\frac{\sqrt{3}}{2},\frac{1}{2}\right\rangle \cdot 10\langle 1,0\rangle}{|10\langle 1,0\rangle|} = 25\sqrt{3}$$

It follows that the total work done, which is the product of the magnitudes of the horizontal force and displacement vectors, is

$$|proj_d\,\boldsymbol{F}||\boldsymbol{d}| = 250\sqrt{3}$$