Chapter 11: Double Integral

Double Integral

In one-variable calculus, we have known that integration can be used to evaluate the area of the graph of a given function, the technique is called definite integral. Naturally, we will extend to multiple variable integration from this topic, simply call multiple integration. Before going into details, we recap certain terminologies to facilitate the subsequent discussions as well as notations.

Consider

$$\int_a^b f(x)\, dx$$

We call

1. a and b to be lower and upper limit respectively.

2. The function $f(x)$ to be integrated is named as integrand.

3. The symbol dx separated from the integrand by some spaces is named as differential.

For this integral, we say that we are integrating the function $f(x)$ over the interval $a \leq x \leq b$.

One should be aware that the techniques learnt previously will be directly applied in this topic.

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We will attempt to generalize the ideas of multiple integration which plays significant role in engineering applications. As a matter of fact, evaluating multivariable integral is somewhat similar to that in single variable case. You will often see geometrical interpretation would be quite helpful to set up and tackle the multivariable integral with ease.

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Double integrals in Rectangular Coordinates

We will start with discussion of geometric meaning of double integrals. Consider the following integrals.

$$\int_{x=a}^{x=b} \int_{y=c}^{y=d} f(x,y)\,dy\,dx$$

For brevity, we call

$$A(x) = \int_{y=c}^{y=d} f(x,y)\,dy$$

to be inner integral. And,

$$\int_{x=a}^{x=b} A(x)dx$$

is called overall integral.

The inner integral is an integral with respect to (w. r. t., in short) y with x being unchanged. Mathematically, $A(x)$ represents the area under the curve by moving along the y axis direction from $y = c$ to $y = d$ over the surface $z = f(x,y)$ while keeping x fixed.

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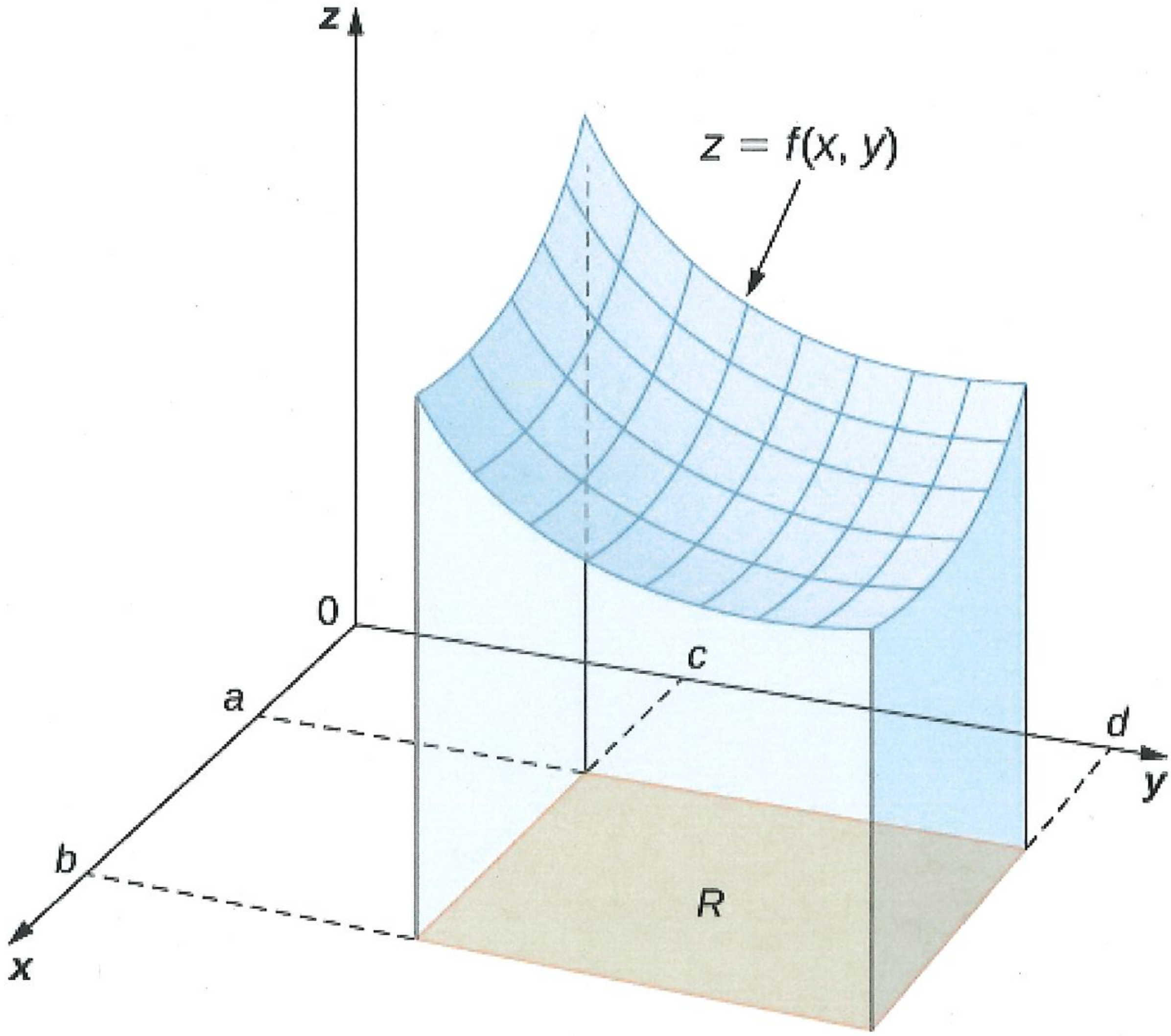
See the following figure.



On the other hand, the overall integral integrates the inner one along the x axis direction from $x = a$ to $x = b$ i.e.

$$\int_{x=a}^{x=b} A(x)dx.$$

Product of $A(x)$ and dx, i.e. $A(x)dx$, can be regarded as the volume of a thin slice with cross-section area $A(x)$ and width dx.

Imagine that if we sum up or stack those thin slices together, the overall (double) integral is the volume under $z = f(x, y)$ over the rectangle confined by

$$x = a,$$

$$x = b,$$

$y = c$ and

$$y = d.$$

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Example 1:

Compute the following double integral

$$\int_{x=-5}^{x=4}\int_{y=0}^{y=3}(2x-4y^3)\,dy\,dx$$

$$=\int_{-5}^{4}(2xy-y^4)\,|_0^3\;dx$$

$$=\int_{-5}^{4}(6x-81)\;dx$$

$$=-756$$

From this simple example, we see that the technique studied in single variable calculus will simply and directly be applied.

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One may ask that since double integral accounts for the volume of a solid there should be no difference if we sum up or stack the thin slices in a different direction or order for the same solid. Actually, it is correct! See the following:

Example 2:

By previous example , we re-compute the integral in the following manner

$$\int_{y=0}^{y=3} \int_{x=-5}^{x=4} (2x - 4y^3)\, dx\ dy$$

It comes as no surprise that the answers of Examples 1 & 2 are identical.

Geometrically, it follows that the volume is the same no matter which direction or order the integration is taken. Theoretically, the following theorem called **Fubini's Theorem** asserts that changing dx and dy (together with respective limits on the integral signs of course!) will give the same result of the double integral.

Theorem 1: Fubini's Theorem for Rectangular Region

Let $f(x,y)$ be a continuous function over a rectangular region with $a \leq x \leq b$ and $c \leq y \leq d$ then

$$\int_a^b \int_c^d f(x,y)\, dydx = \int_c^d \int_a^b f(x,y)\, dx\, dy$$

These integrals are sometimes called <u>iterated integrals</u>.

For simplicity, we may sometimes denote $dxdy$ or $dydx$ as dA and the rectangular region by R. Then, we can write the mentioned integral as

$$\iint_R f(x,y)\, dA$$

where $R = [a,b] \times [c,d]$.

Note also that some books may use different presentations for R. For instance,

$$R = \{(x, y): a \leq x \leq b, c \leq y \leq d\}$$

$$= \{(x, y) \mid a \leq x \leq b, c \leq y \leq d\}$$

Example 3:

Evaluate

$$\iint_R (3x+2y)^{-2}\, dA$$

$$R = [1,2]\times[0,1].$$

Solution:

$$\iint_R (3x+2y)^{-2}\, dA$$

$$= \int_1^2 \int_0^1 (3x+2y)^{-2}\, dy\, dx$$

$$= \int_1^2 (-\frac{1}{2}(3x+2y)^{-1}) \Big|_0^1\, dx$$

$$= \frac{1}{2}\int_1^2 [(3x+2)^{-1} - (3x)^{-1}]\, dx$$

$$= -\frac{1}{6}[\ln|3x+2|\,\Big|_1^2 - \ln|x|\,\Big|_1^2]$$

$$= -\frac{1}{6}(\ln 8 - \ln 5 - \ln 2)$$

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So far, we have discussed double integration over rectangular regions and worked on some examples.

What if the concerned region is no longer rectangular and instead a general (any) region? We will explore this issue soon. To start with we state Fubini's Theorem for general regions.

Theorem 2: Fubini's Theorem for General Regions

Let R be a (general) region over xy-plane and $f(x, y)$ be a continuous function on R, then

$$\iint_R f(x,y)\,dydx = \iint_R f(x,y)\,dxdy$$

Note that the limits of each integral sign have to be matched with the region R.

Example 4:

Evaluate the following double integral

$$\iint_D 3x^2 e^{xy+1}\, dA$$

where

$$D = \{(x, y): 0 \le x \le 2,\ \ x \le 2y \le 2\}.$$

Firstly, we take a look the general region D

$$D = \{(x, y): 0 \le x \le 2,\ \ x \le 2y \le 2\}$$

$$= \{(x, y): 0 \le x \le 2,\ \ \frac{1}{2}x \le y \le 1\}$$

Secondly, we incorporate the region D with the double integral. We have

$$\iint_D 3x^2 e^{xy+1}\, dA$$

$$= \int_{x=0}^{x=2} \int_{y=\frac{x}{2}}^{y=1} 3x^2 e^{xy+1}\, dy\, dx$$

$$= \int_0^2 \int_{\frac{x}{2}}^1 3ex^2 e^{xy}\, dy\, dx$$

$$= 3e \int_0^2 \left(xe^{xy} |_{\frac{x}{2}}^{1} \right) dx$$

$$= 3e \int_0^2 \left(xe^x - xe^{\frac{x^2}{2}} \right) dx$$

$$= 3e \left(xe^x - e^x - e^{\frac{x^2}{2}} \right) |_0^2$$

$$= 6e$$

Example 5:

Evaluate the following double integral

$$\int_0^1 \int_y^1 2x^{-1} \cos x \ dxdy.$$

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Double Integrals in Polar Coordinates

In previous parts, we centered on the double integral over Cartesian rectangular coordinates. If the region of integration is circular in shape, like a disk or ring or integrand exhibiting the feature of rotational symmetry, it is more handy to employ polar coordinates to establish the integrals than the Cartesian one.

For example, consider the following double integral

$$\iint_D e^{x^2} e^{y^2} \, dA$$

D is a unit circle centered at the origin.

Polar coordinate can provide us with very convenient computational environment to solve this kind of problem.

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Before working on some examples, we need to carefully study the structure of double integral using polar coordinates. The polar coordinates depend on 2 variables, r and θ, For a double integral in polar coordinates of a region $a \leq r \leq b$ and $\alpha \leq \theta \leq \beta$. We set up the limits of integral signs as

$$\int_{\theta=\alpha}^{\theta=\beta} \int_{r=a}^{r=b}$$

or

$$\int_{r=a}^{r=b} \int_{\theta=\alpha}^{\theta=\beta}.$$

subject to the order of integration $drd\theta$ or $d\theta dr$.

One thing we have to be very careful is that, in rectangular coordinates, we have $dA = dxdy$ or $dydx$ but in polar coordinates, $dA = rdrd\theta$ or $dA = rd\theta dr$.

Example 6:

Evaluate

$$\iint_D e^{x^2} e^{y^2} \, dA$$

where D is a unit circle centered at the origin.

Firstly, we want to define the region D.

$$0 \le \theta \le 2\pi, \qquad 0 \le r \le 1$$

In term of polar coordinates, $x = r\cos\theta$ and $y = r\sin\theta$.

Then the required integral becomes

$$\begin{aligned}\iint_D e^{x^2} e^{y^2} \, dA &= \int_0^{2\pi} \int_0^1 e^{r^2\cos^2\theta} e^{r^2\sin^2\theta} \, r dr d\theta \\ &= \int_0^{2\pi} \int_0^1 r e^{r^2} dr d\theta \\ &= \int_0^{2\pi} \frac{e-1}{2} \, d\theta \\ &= (e-1)\pi\end{aligned}$$

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Calculation of Area using Double Integral

Throughout the context of Double integral so far, it has centered on calculating volume. But the double integral can in fact be taken to calculate area by considering the double integral

$$\iint dA$$

In the case, we can consider the typical double integral $\iint f(x,y)dA$ by setting $f(x,y) = 1$.

We can interpret that the volume V of a right cylinder in a region R with cross-sectional area A of the region R and height h is equal to $V = A \cdot h$

Now, we consider the height of the cylinder is 1 unit. Thus, it follows that

$$V = \iint_R 1 \cdot \, dA = (\text{area of } R) \cdot 1,$$

and we re-write as

$$area\ of\ R = \iint_R dA$$

Please be careful that this formula is just intended to evaluate the numerical value of the area (or volume) but not the unit since they are clearly not compatible!

Example 7:

Find the area of a region R bounded by $y = \frac{x^2}{2}$ and $y = 2x$

We need to solve for the intersection point(s) of these curves by solving

$$\begin{cases} y = 2x \\ y = \dfrac{x^2}{2} \end{cases}$$

We have $(0,0)$ and $(4,8)$.

Thus, the required area of R is given by

$$\iint_R dA = \int_0^4 \int_{\frac{x^2}{2}}^{2x} dy\, dx$$

$$= \int_0^4 \left(2x - \frac{x^2}{2}\right) dx$$

$$= \frac{16}{3}$$

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Change of Variables

In previous parts, we have dealt with a case regarding change of variables in multiple integrals, say from variables of rectangular coordinates to that of polar coordinates in double integral. In this part, we attempt to generalize the way we carry out the "transformation". We start with introduction of Jacobian Transformation for the double integrals.

Definition 1

The Jacobian of the 2-variable transformation

$$x = x(u, v) \ \& \ y = y(u, v)$$

is given by

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

The Jacobian is defined as a 2 by 2 determinant.

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Theorem 1

Suppose we integrate a function $f(x, y)$ over a region R. Under the transformation (change of variables) $x = x(u, v), y = y(u, v)$, the region becomes S and the integral becomes

$$\iint_R f(x, y) dA_{xy} = \iint_S f(x(u, v), y(u, v)) \left|\frac{\partial(x, y)}{\partial(u, v)}\right| dA_{uv}$$

Note that:

i) We denote dA_{uv} in the integral for variables u and v which is in terms of du and dv after we finish conversion of variables so as not to confuse with the old notation of dA which is about dx and dy. Anyway, we generally write dA for both cases.

ii) We take an absolute value for the Jacobian, i.e.

$$\left|\frac{\partial(x, y)}{\partial(u, v)}\right|$$

iii) For the differential part.

$$dA_{xy} = \left|\frac{\partial(x, y)}{\partial(u, v)}\right| dA_{uv}$$

Example 8:

Show that $dA = r\ drd\theta$ when we change from rectangular coordinates to polar coordinates.

We have the standard conversion formula as follows

$$x = r\cos\theta\ \ \& \ y = r\sin\theta$$

Now, by Definition 1,

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$

$$= \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r(\cos^2\theta + \sin^2\theta) = r$$

Therefore, we have

$$dA_{xy} = \left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| dA_{r\theta}$$

$$dA = r\ dr\ d\theta$$

The result follows.

Example 9:

Evaluate

$$\iint_R 4(x+y)dA$$

where R is the region enclosed by the lines, $y = \pm x$ and $y = \pm(x-5)$, by using the transformation $x = 2u + 3v$ and $y = 2u - 3v$.

Under the suggested transformation, we substitute them into given equations for R.

For $y = x$, we have

$$2u - 3v = 2u + 3v$$
$$v = 0$$

For $y = -x$, we have

$$u = 0$$

For $y = -x + 5$, we have

$$u = \frac{5}{4}$$

For $y = x - 5$, we have

$$v = \frac{5}{6}$$

By the above results, the transformed region S becomes a rectangle enclosed by

$$u = 0, u = \frac{5}{4}$$
$$v = 0 \;\&\; v = \frac{5}{6}.$$

This implies $0 \leq u \leq \frac{5}{4}$ and $0 \leq v \leq \frac{5}{6}$.

We can carry out the Jacobian now

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} 2 & 3 \\ 2 & -3 \end{vmatrix}$$

$$= -12$$

Hence, the required double integral

$$\iint_R 4(x+y)dA = 4\iint_R (2u+3v+2u-3v)\cdot|-12|\,dudv$$

$$= \int_0^{\frac{5}{6}}\int_0^{\frac{5}{4}} 192u\,dudv$$

$$= 150\int_0^{\frac{5}{6}} dv$$

$$= 125.$$